UK Maths Trust

Junior Mathematical Challenge

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Solutions and investigations

25 April 2024

These solutions augment the shorter solutions also available online. The solutions given here are full solutions, as explained below. In some cases we give alternative solutions. There are also many additional problems for further investigation. We welcome comments on these solutions. Please send them to challenges@ukmt.org.uk.

The Junior Mathematical Challenge (JMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that occasionally you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can sometimes be a sensible thing to do in the context of the JMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. Therefore here we have aimed at giving full solutions with all steps explained (or, sometimes, left as an exercise). We hope that these solutions can be used as a model for the type of written solution that is expected when a complete solution to a mathematical problem is required (for example, in the Junior Mathematical Olympiad and similar competitions).

Enquiries about the Junior Mathematical Challenge should be sent to:

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☎ 0113 365 1121 challenges@ukmt.org.uk www.ukmt.org.uk

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
C	B	B	A	E	E	D	C	A	E	B	E	C	E	D	B	C	D	C	E	B	D	D	B	A

1. When the five expressions below are simplified, how many different values are obtained?

$2+2$ $\quad 2\times 2$ $\quad 2-2$ $\quad 2\div 2$ $\quad 2^2$

A 1 B 2 C 3 D 4 E 5

SOLUTION **C**

We have

$$2+2=4, \quad 2\times 2=4, \quad 2-2=0, \quad 2\div 2=1 \text{ and } 2^2=4.$$

Therefore the expressions take 3 different values, 0, 1 and 4.

FOR INVESTIGATION

1.1 When the five expressions below are simplified, how many different values are obtained?

$2^{(2^2)}$ $\quad 2\times 2^2$ $\quad (2^2)^2$ $\quad 2^2\div 2$ $\quad 2^{2+2}$

2. Which of the following could have a capacity of 10 litres?

A An aeroplane B A bucket C A cup D A dustpan E An egg

SOLUTION **B**

Since 1 litre = 1000 cc, 10 litres = 10 000 cc.

Since $20^3 = 8000 < 10\,000 < 30^3 = 27\,000$, we have $20 < \sqrt[3]{10\,000} < 30$. Therefore 10 litres is the volume of a cube with side-length between 20 cm and 30 cm.

We therefore see that the capacity of an aeroplane is very much greater than 10 litres. Also, a cup, a dustpan and an egg have capacities much less than 10 litres. A small household bucket has a capacity of around 10 litres.

FOR INVESTIGATION

2.1 Approximately how many average cups of water are needed to fill a 10 litre bucket?

3. Gill is 36 this year. In which year will her age next be a square?

A 2025 B 2037 C 2047 D 2052 E 2060

SOLUTION **B**

The next square after 36 is 49 $(= 7^2)$. Now $49 - 36 = 13$. Hence Gill will be 49 in 13 years' time, that is, in the year 2037, since $2024 + 13 = 2037$.

FOR INVESTIGATION

3.1 In how many years between now (2024) and 2037 will Gill have prime number birthday (that is, a birthday in which her age becomes a prime number)?

4. A drink is made by mixing one part of cordial with four parts of water. What percentage of the drink is cordial?

A 20 B 25 C 40 D 75 E 80

SOLUTION **A**

The cordial makes up one part in five of the drink.

As a fraction this is $\frac{1}{5}$. Therefore the percentage of the drink that is cordial is

$$\tfrac{1}{5} \times 100\% = 20\%.$$

FOR INVESTIGATION

4.1 A drink is made by mixing one part of cordial with nine parts of water.

What percentage of the drink is cordial?

4.2 A drink is made by mixing one part of cordial with n parts of water. This drink contains 12.5% of cordial.

What is the value of n?

5. What is the value of $1 + 2 - 3 \times 4 \div 5$?

A 0.2 B 0.3 C 0.4 D 0.5 E 0.6

SOLUTION **E**

$$1 + 2 - 3 \times 4 \div 5 = (1 + 2) - ((3 \times 4) \div 5) = 3 - (12 \div 5) = 3 - 2.4 = 0.6.$$

FOR INVESTIGATION

5.1 (a) What is the value of

$$1 \times 2 + 3 \div 4 - 5?$$

(b) What is the value of

$$2 + 3 - 4 \times 5 \div 6\,?$$

5.2 What is the largest number which is the value of the expression

$$1 \,\square\, 2 \,\square\, 3 \,\square\, 4 \,\square\, 5$$

when one of the symbols $+ \;\; - \;\; \times \;\; \div$ is used to replace each $\square$,

(a) if each of the symbols $+ \;\; - \;\; \times \;\; \div$ may be used more than once, and

(b) if each of the symbols $+ \;\; - \;\; \times \;\; \div$ may be used just once?

6. Which of the following has the same remainder when divided by 3 as it does when divided by 4?

A 7 B 11 C 17 D 19 E 25

SOLUTION **E**

The remainders when the given numbers are divided by 3 and by 4 are shown here:

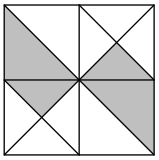
remainder when divided by	7	11	17	19	25
3	1	2	2	1	1
4	3	3	1	3	1

We see that, of the given options, the only number which has the same remainder when divided by 3 and by 4 is 25.

FOR INVESTIGATION

6.1 Which is the smallest positive integer greater than 5 which has the same remainder when divided by 5 as it does when divided by 6?

7. The diagram shows a large square which has been divided into four smaller squares. It also shows both diagonals of the large square and two diagonals of smaller squares.



What fraction of the area of the large square has been shaded?

A $\frac{3}{16}$ B $\frac{1}{4}$ C $\frac{5}{16}$ D $\frac{3}{8}$ E $\frac{7}{16}$

SOLUTION **D**

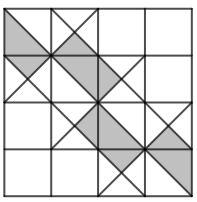
Each smaller square makes up $\frac{1}{4}$ of the large square. One half of two of the smaller squares are shaded, making up one small square. One quarter of each of the other two small squares are shaded, making up half a small square, and hence one eighth of the large square.

Therefore the fraction of the large square that is shaded is

$$\frac{1}{4} + \frac{1}{8} = \frac{3}{8}.$$

FOR INVESTIGATION

7.1 The diagram shows a square which is divided into sixteen smaller squares, all the same size.



What fraction of the larger square is shaded?

8. Skye has half as many pens as Ishaa. Ana has twice as many pens as Skye.

What fraction of all their pens does Skye have?

A $\frac{1}{3}$ B $\frac{1}{4}$ C $\frac{1}{5}$ D $\frac{1}{6}$ E $\frac{1}{8}$

SOLUTION **C**

Let p be the number of pens that Skye has.

Since Skye has half as many pens as Ishaa, Ishaa has $2p$ pens.

Since Ana has twice as many pens as Skye, Ana also has $2p$ pens.

Therefore Skye, Ishaa and Ana between them have $p + 2p + 2p = 5p$ pens.

So the fraction of all the pens that Skye has is

$$\frac{p}{5p} = \frac{1}{5}.$$

FOR INVESTIGATION

8.1 Suppose that Skye has one-third of the number of pens that Ishaa has, and Ana has three times the number of pens that Skye has.

In this case, what fraction of all the pens does Skye have?

8.2 Suppose that Skye has one-quarter of the number of pens that Ishaa has, and Ana has five times the number of pens that Skye has.

In this case, what fraction of all the pens does Skye have?

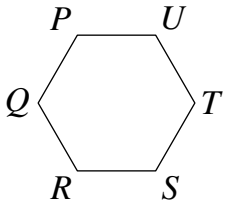
8.3 Suppose that Ana has six times the number of pens that Skye has, and that Skye has one tenth of all the pens that Skye, Ishaa and Ana have between them.

In this case, what fraction of the number of pens that Ishaa has does Skye have?

9. The diagram shows the regular hexagon $PQRSTU$.

What the size of angle UPT?

A 30° B 45° C 60° D 120° E 150°



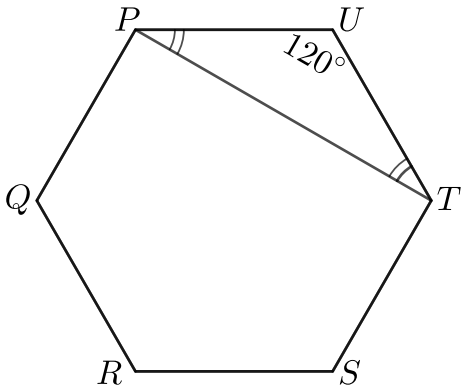
SOLUTION **A**

The interior angles of a regular hexagon are all 120°. Therefore $\angle TUP = 120°$.

The angles of a triangle add to 180°. Therefore, from the triangle UPT, we have

$$\angle UPT + \angle UTP + 120° = 180°. \qquad (1)$$

Because $PQRSTU$ is a regular hexagon, $PU = TU$. Therefore the triangle PTU is isosceles. Hence $\angle UPT = \angle UTP$.



It follows from (1) that

$$2 \times \angle UPT + 120° = 180°.$$

Therefore

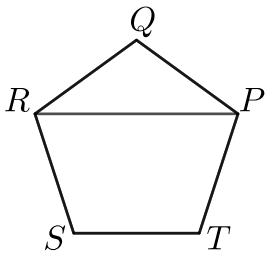
$$\angle UPT = \tfrac{1}{2}(180° - 120°) = \tfrac{1}{2}(60°) = 30°.$$

FOR INVESTIGATION

9.1 Prove that the interior angles of a regular hexagon are all 120°.

9.2 $PQRST$ is a regular pentagon.

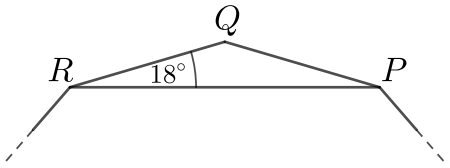
What is the angle QRP?



9.3 P, Q and R are adjacent vertices of a regular polygon.

$\angle QRP = 18°$.

How many vertices does the polygon have?



10. In Fred's field there are some humans and some horses.
There are 25 heads in total and 60 legs in total.

What is the difference between the number of humans and the number of horses?

A 3 B 7 C 9 D 11 E 15

SOLUTION **E**

Humans and horses each have one head. So the total number of humans and horses in the field is 25.

Humans have 2 legs and horses have 4 legs. If there were 25 humans in the field, the number of legs would be $25 \times 2 = 50$. This is 10 fewer than the 60 legs that are in the field.

These 10 extra legs must arise because there are some horses in the field. Each horse has 2 more legs than a human. So the 10 extra legs come from $10 \div 2 = 5$ horses.

So there are 5 horses in the field. Hence the number of humans in the field is $25 - 5 = 20$,

We can now deduce that the difference between the number of humans and the number of horses in Fred's field is $20 - 5 = 15$.

FOR INVESTIGATION

10.1 Check that with 20 humans and 5 horses in the field, there are 25 heads and 60 legs in the field.

10.2 In this problem we discuss an alternative approach to Question 10 which uses simultaneous linear equations. If you have not yet met this topic in algebra, please go straight to Problem 10.3.

Suppose that there are x humans and y horses. Because there are 25 heads in total,

$$x + y = 25. \qquad (1)$$

Because there are 60 legs in total,

$$2x + 4y = 60. \qquad (2)$$

(a) Solve the simultaneous linear equations (1) and (2) to find the values of x and y. Hence calculate the value of $x - y$.

(b) x and y satisfy the simultaneous linear equations

$$x + y = 40,$$

and

$$2x + 4y = 110.$$

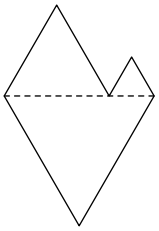
Find the value of $x - y$ directly, without first finding the separate values of x and y.

10.3 In Cate's Café there are some three-legged tables and some four-legged chairs.

The total number of tables and chairs is 50. Between them these tables and chairs have 188 legs.

How many tables were there in Cate's Café?

11. A hexagon is formed by arranging three equilateral triangles, as shown in the diagram.
The side-length of the largest equilateral triangle is 10 cm.

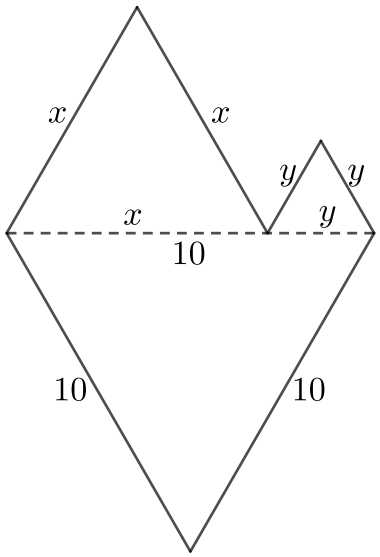


What is the perimeter, in cm, of the hexagon?

A 45 B 40 C 35 D 30
E more information needed

SOLUTION **B**

Let the side-lengths, in cm, of the two smaller equilateral triangles be x and y, as shown in the diagram.



We see from the diagram that the sum of the side-lengths of the two smaller equilateral triangles is equal to the side-length of the largest equilateral triangle. That is, $x + y = 10$.

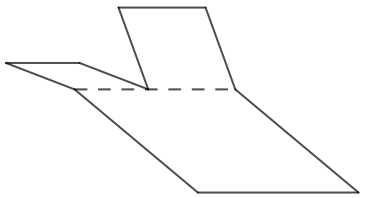
Therefore the perimeter, in cm, of the hexagon is

$$\begin{aligned} 10 + 10 + x + x + y + y &= 20 + 2(x + y) \\ &= 20 + 2 \times 10 \\ &= 20 + 20 \\ &= 40. \end{aligned}$$

Note: Note that this answer does not depend on the values of x and y but only on the fact that $x + y = 10$.

FOR INVESTIGATION

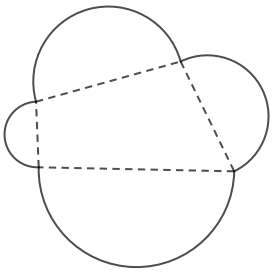
11.1 The diagram shows a shape made of three rhombuses joined together.



The largest rhombus has a perimeter of length 36 cm.

What is the perimeter of the whole shape?

11.2 The diagram shows a shape whose boundary is made up of four semicircles.



The diameters of these semicircles form a quadrilateral.

The perimeter of this quadrilateral has length 25 cm.

What is the perimeter of the shape?

12. In the multiplication sum shown, y represents the same digit each time.

What is the value of y?

A 0 B 1 C 4 D 5 E 6

$$\begin{array}{r} 3\ y \\ \times\ \ 4\ y\ y \\ \hline 1\ y\ 7\ 7\ y \end{array}$$

SOLUTION **E**

From the units digits we see that y is a digit with the property that y^2 has y as its units digit. Therefore y is 0, 1, 5 or 6.

Now '$3y$' ≥ 30 and '$4yy$' ≥ 400. Therefore '$1y77y$' $\geq 30 \times 400 = 12\,000$. Hence $y \neq 0$ and $y \neq 1$.

It remains only to check the values $y = 5$ and $y = 6$.

METHOD 1

We have $35 \times 455 = 15\,925$ and $36 \times 466 = 16\,776$. It follows that $y = 6$.

METHOD 2

35 is divisible by 7, but 15 775 is not divisible by 7. Therefore $35 \times 455 \neq 15\,775$. Hence $y \neq 5$. In the context of the JMC, we can conclude that $y = 6$.

FOR INVESTIGATION

12.1 Check that 15 775 is not divisible by 7.

12.2 In the multiplication sum

$$\begin{array}{r} 3\ y \\ \times\ \ 4\ 4\ y \\ \hline 1\ y\ y\ 7\ y \end{array}$$

y is a non-zero digit. Find its value.

12.3 In the multiplication sum

$$\begin{array}{r} 8\ y \\ \times\ \ x\ y\ y \\ \hline y\ y\ x\ 7\ y \end{array}$$

x and y are non-zero digits. Find their values.

12.4 In the multiplication sum

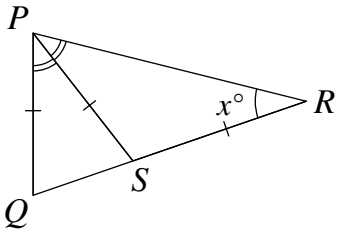
$$\begin{array}{r} y\ y \\ \times\ \ x\ y\ y \\ \hline y\ z\ y\ z\ y \end{array}$$

x, y and z are non-zero digits. Find their values.

13. In the triangle PQR, the point S is on the edge QR. $\angle QPS = \angle SPR$, $PQ = PS = SR$ and $\angle PRQ = x°$.

What is the value of x?

A 30 B 33 C 36 D 40 E 45



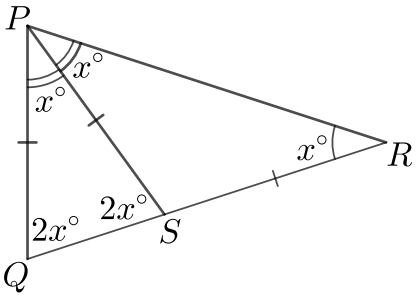
SOLUTION **C**

Since $PS = SR$, the triangle SRP is isosceles and therefore $\angle SPR = \angle PRS = \angle PRQ = x°$.

Hence, by the External Angle Theorem [see Problem 13.1], $\angle QSP = \angle SPR + \angle PRS = x° + x° = 2x°$.

Since $PQ = PS$, the triangle PQS is isosceles and therefore $\angle SQP = \angle QSP = 2x°$.

Also, $\angle QPS = \angle SPR = x°$.



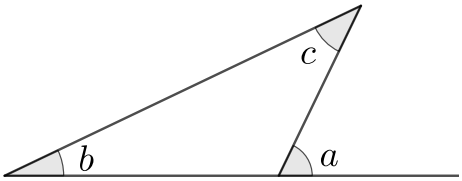
We now apply the fact that the sum of the angles in a triangle is 180° to the triangle PQS. This gives $x + 2x + 2x = 180$. Hence $5x = 180$. It follows that $x = 36$.

FOR INVESTIGATION

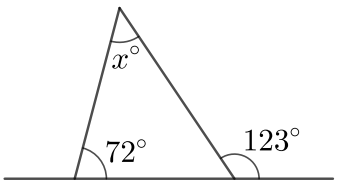
13.1 The *External Angle Theorem* says that the external angle of a triangle is the sum of the two opposite internal angles.

In terms of the diagram it says that $a = b + c$.

Prove the External Angle Theorem.



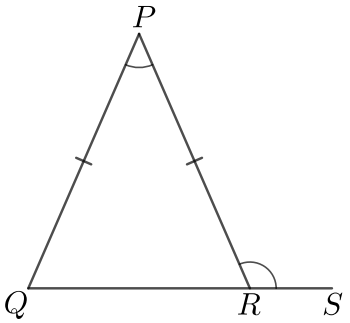
13.2 Find the value of x in the triangle shown in the diagram on the right.



13.3 In the diagram shown on the right PQR is an isosceles triangle in which $PQ = PR$.

QRS is a straight line and $\angle SRP = 5 \times \angle RPQ$.

Find all the angles in the triangle PQR.



14.

		÷		+		=	

The digits 1, 2, 3, 5 and 8 are to be placed in the grid above, one to a cell, to make a correct mathematical statement.

Which number should come immediately after the division sign?

A 1 B 2 C 3 D 5 E 8

SOLUTION **E**

We let w be the two-digit number that occurs before the ÷ sign, and x, y and z be the one-digit numbers that occur in the positions shown.

w	÷	x	+	y	=	z

We have $w \div x + y = z$. Hence $w \div x = z - y$. Since y and z are integers, it follows that $w \div x$ is an integer. Therefore w is a multiple of x.

We also note that as $w \div x > 0$, we have $y < z$.

Also, $y \geq 1$ and $z \leq 8$. Hence $w \div x \leq 8 - 1 = 7$. Since $w \geq 12$, it follows that $x \neq 1$.

Suppose $x = 2$. Then w is a multiple of 2. Hence the units digit of w has to be 8 and $w \geq 18$. So $w \div x \geq 18 \div 2 = 9$ which we have seen is not possible. Hence $x \neq 2$.

Suppose $x = 3$. Then w is a multiple of 3. Since $w \div x \leq 7$, it follows that $w \leq 21$. Because w can only use the digits 1, 2, 5 and 8, w is either 12, 15, 18 or 21.

If $w = 12$, we would have $w \div x = 12 \div 3 = 4$. The sum would then be $4 + 5 = 8$, which is not true. Similarly, if $w = 15$, the sum would be $5 + 2 = 8$, which again is not true. Likewise, If $w = 18$, the sum would be $6 + 2 = 5$, which is also not true. Finally, if $w = 21$, the sum would be $7 + 5 = 8$ Again, this is not true. Hence $x \neq 3$.

If $x = 5$, then w would be a multiple of 5 and would consist of two of the digits 1, 2, 3 and 8. This is impossible. So $x \neq 5$.

This leaves only the case $x = 8$. Then w is a multiple of 8 which consists of two of the digits 1, 2, 3 and 5. The only possibility is $w = 32$ so that $w \div x = 32 \div 8 = 4$. The remaining digits are 1 and 5. We therefore see that the correct mathematical statement is $32 \div 8 + 1 = 5$.

Hence the digit 8 should come immediately after the division sign.

FOR INVESTIGATION

14.1

		÷		+		=	

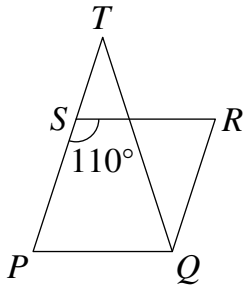
(a) Place the digits 1, 5, 6, 7 and 9 in the grid above, one to a cell, to make a correct mathematical statement.

(b) Place the digits 2, 3, 4, 6 and 7 in the grid above, one to a cell, to make a correct mathematical statement.

15. In the diagram shown, $PQRS$ is a rhombus and PQT is an isosceles triangle in which $PT = QT$. Angle $PSR = 110°$.

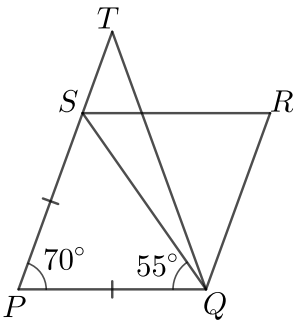
What is the size of angle SQT?

A 5° B 10° C 12.5° D 15° E 20°



SOLUTION **D**

We have $SP = PQ$. Therefore the triangle SPQ is isosceles, and $\angle PQS = \angle PSQ$.



The angles in the triangle SPQ have sum 180°. Therefore

$$\angle PSQ + \angle PQS + 70° = 180°.$$

Hence

$$2 \times \angle PQS + 70° = 180°.$$

Therefore

$$\angle PQS = \tfrac{1}{2}(180° - 70°) = 55°.$$

Because $PQRS$ is a rhombus, SR is parallel to PQ. Therefore $\angle PSR + \angle SPQ = 180°$. Hence $\angle SPQ = 180° - 110° = 70°$.

Because $PT = QT$, the triangle PTQ is isosceles and so $\angle TQP = \angle TPQ = \angle SPQ = 70°$.

It follows that

$$\angle SQT = \angle TQP - \angle PQS = 70° - 55° = 15°.$$

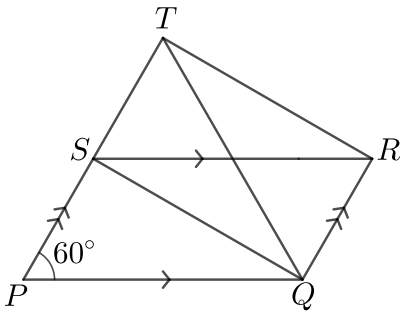
FOR INVESTIGATION

15.1 $PQRS$ is a parallelogram in which PQ is twice the length of PS.

PST is a straight line.

TPQ is an isosceles triangle in which $TP = TQ$ and $\angle QPT = 60°$.

Prove that $QRTS$ is a rectangle.



16. The world's smallest vertebrate is much shorter than its name! Discovered in 2022, the frog *paedophryne amauonsis* is only 7.7 mm long.

Approximately how many of these frogs, placed end to end, would be needed to make a line 1 metre long?

A 100 B 130 C 260 D 390 E 520

SOLUTION **B**

Since 1 metre = 1000 mm, we need to find which of the given options gives the closest approximation to the value of $\frac{1000}{7.7}$.

METHOD 1

The number 7.7 rounds up to 8. Therefore the value of $\frac{1000}{8}$ slightly underestimates the value of $\frac{1000}{7.7}$. Since $\frac{1000}{8} = 125$, we can conclude that approximately 130 of the frogs would be needed to make a line 1 metre long.

METHOD 2

We exploit the fact that $1001 = 7 \times 11 \times 13$.

Using this fact, we have $\frac{1000}{7.7} \approx \frac{1001}{7.7} = \frac{7 \times 11 \times 13}{7 \times 1.1} = 10 \times 13 = 130$.

FOR INVESTIGATION

16.1 The bee hummingbird, *Mellisuga helenae*, is native to the island of Cuba. It is the smallest known bird. Female bee hummingbirds weigh an average of 74 grams. The males are lighter with an average weight of 55 grams.

Approximately how many female bee hummingbirds would weigh 1 kilogram and approximately how many male bee hummingbirds would weigh 1 kilogram?

16.2 (a) Check that $7 \times 11 \times 13 = 1001$.

(b) What is the value of $347 \times 7 \times 11 \times 13$?

(c) Write down a three-digit number. Turn it into a six-digit number by repeating the same three digits. (For example, if you wrote down the number 123, you should now have the six-digit the number 123123.)

Suppose you now

(i) divide your six-digit number by 7 to obtain your first answer, next

(ii) divide your first answer by 11 to obtain your second answer, and finally

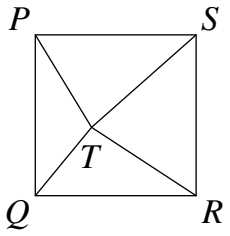
(iii) divide your second answer by 13 to obtain your final answer.

Can you predict what your final answer will be without doing the divisions?

17. $PQRS$ is a square with area $100\,\text{cm}^2$. The point T is inside the square.
QRT is a triangle with area $24\,\text{cm}^2$.

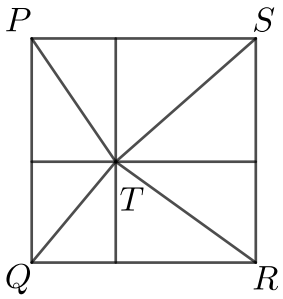
What is the area, in cm^2, of the triangle PTS ?

A 24 B 25 C 26 D 27 E 28



SOLUTION **C**

The diagram on the right shows the lines through T parallel to the edges of the square.



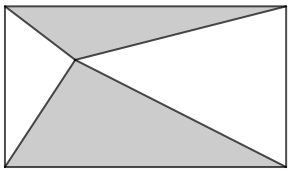
These lines divide the square into four rectangles. The triangles QRT and PTS between them occupy a half of each of these rectangles. Therefore their total area is half the area of the square, that is, $50\,\text{cm}^2$.

The triangle QRT has area $24\,\text{cm}^2$. Hence the area of the triangle PTS is $(50 - 24)\,\text{cm}^2 = 26\,\text{cm}^2$.

FOR INVESTIGATION

17.1 The diagram shows two shaded and two unshaded triangles inside a rectangle.

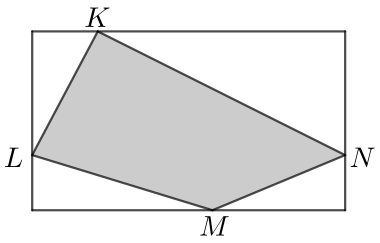
Prove that the total area of the two shaded triangles is equal to the total area of the unshaded triangles.



17.2 The diagram shows a quadrilateral $KLMN$ whose vertices lie on the edges of a rectangle.

The line LN is parallel to the longer sides of the rectangle.

Prove that the area of the quadrilateral $KLMN$ is half of the area of the rectangle.



18. Goldilocks eats three equal-sized bowls of porridge, one after the other.

When she has eaten $\frac{3}{7}$ of the total amount of porridge, what fraction of the porridge in the second bowl has she eaten?

A $\frac{2}{63}$ B $\frac{1}{7}$ C $\frac{1}{3}$ D $\frac{2}{7}$ E $\frac{1}{2}$

SOLUTION **D**

Let x be the fraction of the second bowl that Goldilocks has eaten.

Each bowl makes up one-third of the porridge. Therefore the fraction of all the porridge that Goldilocks has eaten is $\frac{1}{3} + \frac{1}{3}x$. Therefore

$$\frac{1}{3} + \frac{1}{3}x = \frac{3}{7}.$$

We multiply both sides of this equation by 21. This gives

$$7 + 7x = 9.$$

Hence $7x = 2$ and therefore $x = \frac{2}{7}$.

FOR INVESTIGATION

18.1 When Goldilocks has eaten one-third of the porridge in the second bowl, what fraction of all the porridge has she eaten?

19. Jokers always lie.
Clowns always tell the truth.
A group of four, each of whom is a Joker or a Clown, make the following statements about each other:
P says, "Q always lies"; Q says, "R always lies";
R says, "P always tells the truth"; S says, "Exactly two of P, Q and R are Jokers".

How many of P, Q, R and S are Clowns?

A 0 B 1 C 2 D 3 E 4

SOLUTION **C**

Suppose P is a Clown and hence is telling the truth. So Q is lying and hence is a Joker, R is telling the truth and so is a Clown, and S is telling a lie and hence is a Joker.

Therefore, in this case, there are two Clowns.

Now suppose P is a Joker and hence is telling a lie. Then Q always tells the truth and hence is a Clown, R is telling a lie and hence is a Joker, and S is telling the truth and hence is a Clown.

Therefore, in this case, there are two Clowns.

It follows that in any case there are two Clowns.

20. When you cut a regular hexagon into two pieces with a single straight cut, you get two polygons.

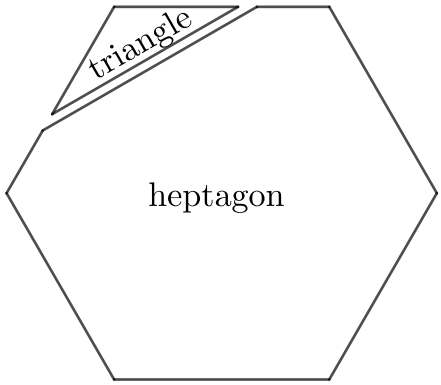
Which of these shapes cannot be obtained?

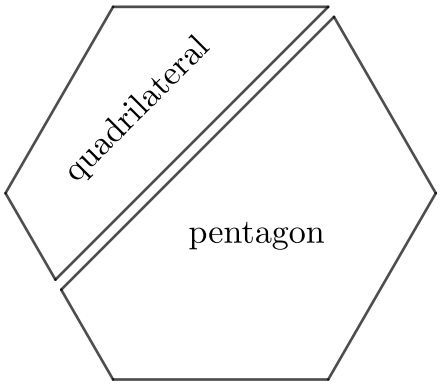
A A triangle B A quadrilateral C A pentagon
D A heptagon E An octagon

SOLUTION **E**

When a regular hexagon is cut by a single straight cut, the cut forms one edge of both of the resulting polygons. Their other edges are parts or all of some or all of the six edges of the hexagon. Therefore these polygons have at most seven edges. It follows that an octagon cannot be obtained.

Note: In the context of the JMC this is enough to pick out E as the correct option. For a complete answer it would be necessary to show that the other shapes can be obtained. The diagrams below show this.



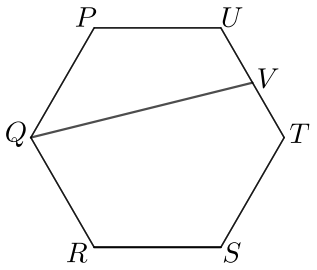


FOR INVESTIGATION

20.1 $PQRSTU$ is a regular hexagon.

V is the midpoint of TU.

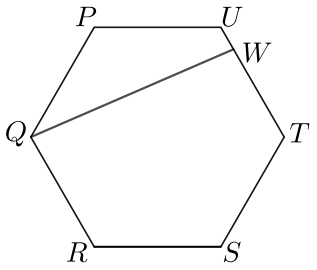
Prove that the area of the quadrilateral $PQVU$ is half the area of the pentagon $QRSTV$.



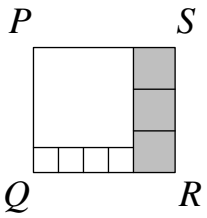
20.2 $PQRSTU$ is a regular hexagon.

W is the point on TU such that the area of the quadrilateral $PQWU$ is one quarter of the area of the pentagon $QRSTW$.

What is the ratio $UW : WT$?



21. The rectangle $PQRS$ is divided into eight squares: one large unshaded square, four small unshaded squares and three shaded squares, as shown in the diagram.



What fraction of the area of rectangle $PQRS$ is shaded?

A $\frac{1}{5}$ B $\frac{5}{17}$ C $\frac{2}{7}$ D $\frac{3}{10}$ E $\frac{1}{3}$

SOLUTION **B**

We let x be the side-length of the four small unshaded squares and let y be the side-length of the three shaded squares.

It follows that the rectangle $PQRS$ has width $4x + y$ and height $3y$.

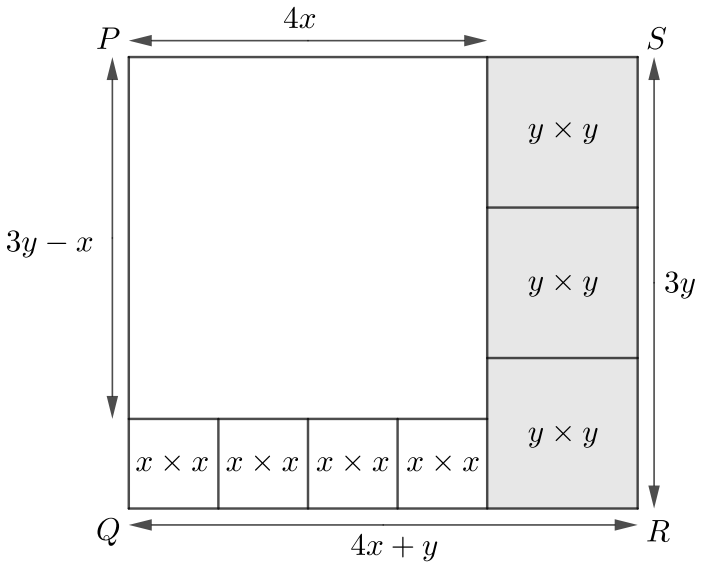
Also, the large unshaded square has width $4x$ and height $3y - x$.

Because it is a square,

$$4x = 3y - x$$

and hence

$$5x = 3y.$$



Therefore $x = \frac{3}{5}y$. It follows that

$$4x + y = 4 \times \tfrac{3}{5}y + y = \tfrac{12}{5}y + y = \tfrac{17}{5}y.$$

Hence the area of the rectangle $PQRS$ is $\frac{17}{5}y \times 3y$.

The three shaded squares form a rectangle with width y and height $3y$. Therefore the area of the rectangle $PQRS$ that is shaded is $y \times 3y$.

It follows that the fraction of the area of the rectangle $PQRS$ that is shaded is

$$\frac{y \times 3y}{\frac{17}{5}y \times 3y} = \frac{y}{\frac{17}{5}y} = \frac{1}{\frac{17}{5}} = \frac{5}{17}.$$

FOR INVESTIGATION

21.1 What is the area of the large unshaded square as a fraction of the area of the rectangle $PQRS$?

21.2 What is the ratio of the area of a small unshaded square to the area of a shaded square?

22. Forty furry ferrets weigh the same as fifty fit ferrets. Forty-five fit ferrets weigh the same as fifty-four friendly ferrets.

How many friendly ferrets weigh the same as fifty furry ferrets?

A 40 B 55 C 60 D 75 E 80

SOLUTION **D**

Let the weight, in some suitable unit, of a furry ferret, a fit ferret and a friendly ferret be x, y and z, respectively.

From what we are told in the question $40x = 50y$ and $45y = 54z$. Therefore $x = \dfrac{50}{40}y = \dfrac{5}{4}y$ and $y = \dfrac{54}{45}z = \dfrac{6}{5}z$.

It follows that

$$50x = 50 \times \frac{5}{4}y = 50 \times \frac{5}{4} \times \frac{6}{5}z = 75z.$$

Hence 75 friendly ferrets weigh the same as fifty furry ferrets.

23. The area of a square is six times the area of a rectangle with a length half that of the square and a width 6 cm less than the width of the square.

What is the perimeter of the square?

A 24 cm B 28 cm C 32 cm D 36 cm E 40 cm

SOLUTION **D**

Let the side length of the square be x cm.

Then the rectangle has length $\frac{1}{2}x$ cm and width $(x - 6)$ cm. It follows that that the area of the rectangle is $\frac{1}{2}x(x - 6)$ cm^2.

Since the area of the square is six times the area of the rectangle, $x^2 = 6\left(\frac{1}{2}x(x - 6)\right)$.

Therefore $x^2 = 3x^2 - 18x$.

Hence $2x^2 = 18x$. Since $x \neq 0$, we can divide both sides of this equation by $2x$. It follows that $x = 9$.

Since the square has side length 9 cm its perimeter is 4×9 cm $= 36$ cm.

FOR INVESTIGATION

23.1 The area of a square is seven times the area of a rectangle with a length one third that of the square and a width 7 cm less than the width of the square.

What is the perimeter of the square?

24. Rovers, United, City and Wanderers played against each other once in a hockey tournament.

The results table is shown below.

Team	Win	Draw	Loss	Goals for	Goals against
Rovers	3	0	0	5	0
United	0	2	1	3	6
City	1	1	1	4	4
Wanderers	0	1	2	0	2

What was the score in the match between Rovers and United?

A 3 - 1 B 3 - 0 C 2 - 1 D 2 - 0 E 1 - 0

SOLUTION **B**

Note first that United drew two of their matches and lost one match. Also they scored three fewer goals than were scored against them.

In the matches they drew they will have scored the same number of goals as their opponents. Hence, in the match which they lost, United lost by a margin of three goals.

Rovers won all their matches. Hence Rovers beat United. Also, Rovers had no goals scored against them.

Therefore United lost to Rovers by a margin of three goals and without scoring a goal.

It follows that the score in the match between Rovers and United was a 3 - 0 win for Rovers.

FOR INVESTIGATION

24.1 Use the information given in this question to fill in the table below to show the number of goals scored by each team in each match.

		Goals for			
		R	U	C	W
Goals against	Rovers	×			
	United		×		
	City			×	
	Wanderers				×

24.2 What is the smallest number of additional goals City would have needed to score in order to win the tournament? [Assume that the teams are awarded 2 points for a win and 1 for a draw, and that a tie is settled by goal difference.]

25. In a school, one fifth of the students have blue eyes. One tenth of the left-handed students have blue eyes. One quarter of the right-handed students have blue eyes.

What fraction of the students are left-handed?

A $\frac{1}{3}$ B $\frac{1}{4}$ C $\frac{1}{5}$ D $\frac{1}{8}$ E $\frac{1}{10}$

SOLUTION **A**

Suppose that that l is the number of left-handed students and r is the number of right-handed students.

Then the total number of students is $l + r$, and the fraction that are left-handed is $\frac{l}{l+r}$.

There are $\frac{1}{10}l$ left-handed students with blue eyes and $\frac{1}{4}r$ right-handed students with blue eyes.

Since one fifth of all the students have blue eyes

$$\frac{1}{10}l + \frac{1}{4}r = \frac{1}{5}(l + r).$$

By multiplying both sides of this equation by 20, we obtain

$$2l + 5r = 4(l + r).$$

That is,

$$2l + 5r = 4l + 4r.$$

Therefore

$$r = 2l.$$

Therefore the fraction of the students who are left-handed is given by

$$\frac{l}{l+r} = \frac{l}{l+2l} = \frac{l}{3l} = \frac{1}{3}.$$

FOR INVESTIGATION

25.1 In another school, one-sixth of the left-handed pupils have blue eyes and eight-ninths of the pupils are right-handed.

Altogether, one quarter of the pupils have blue eyes.

What fraction of the right-handed pupils in this school have blue eyes?